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Piecewise deterministic dynamics from the application of noise to singular equations of motion

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Abstract. We describe a form of *piecewise deterministic* dynamics, where solutions evolve deterministically throughout most of phase space, but, in the presence of noise, make non-deterministic jumps to other solutions when the trajectory passes near a singularity in the equations of motion. The type of singularity we consider in this paper is a single point where the Lipschitz conditions fail and many closed-loop trajectories share a common tangent point. It is shown that there is a finite uncertainty associated with the behaviour that is independent of the magnitude of the noise. The long-term behaviour, while similar in appearance to deterministic chaos, has rather different implications for prediction and control.

1. Introduction

It is often assumed that classical physical systems governed by differential equations are *deterministic* [1], meaning that the forward time evolution is uniquely determined by the current state of the system. Correspondingly, determinism also implies that the system's current state uniquely determines its past behaviour. These qualities follow directly from the *existence and uniqueness theorem* of differential equations [2], which requires that the equations of motion of the system are Lipschitz continuous. However, there is nothing in classical mechanics that *requires* Lipschitz continuity. Indeed, in the case of a cracking whip, the physical solutions imply violation of the Lipschitz condition [3]. A similar effect is seen in seismic waves as they approach the surface of the Earth [4].

If one relaxes the Lipschitz condition, an intriguing possibility arises, namely that uniqueness does not necessarily hold for solutions to the equations of motion. In such a case, solutions may actually intersect and it is not difficult to see that any sort of random fluctuations near such an intersection could have a profound impact on the time evolution of the system. We term such systems *non-deterministic*, as the forward (or reverse) time evolution is not uniquely determined by the equations of motion alone. The *terminal dynamics* described by Zak [5] utilizes such a mechanism where multiple trajectories intersect at a common equilibrium point in finite time. Chen [6] has independently suggested the same behaviour under the heading of *noise-induced instability* (though non-determinism as such is never explicitly mentioned). Other aspects of non-deterministic systems, especially in the presence of noise, have been explored by Hübler [7], and it has also been suggested that non-determinism may play an important role in biological systems [8]. The type of system we shall consider in this paper is of a somewhat different

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flavour. Rather than have the singularity be an equilibrium point, we allow uniqueness to fail at a point where the equations of motion are non-zero. In particular, we examine the case where solutions of the system form a family of closed loops, all sharing a single common tangent point.

2. The non-deterministic harmonic oscillator (NDHO)

We begin with the simple harmonic oscillator (SHO), described by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x = y \qquad \frac{\mathrm{d}}{\mathrm{d}t}y = -x\,. \tag{1}$$

Solutions of (1) are circles in the (x, y) phase plane. The SHO is a deterministic system, so that every point (x, y) belongs to a unique solution described by a circle with a particular radius $r = \sqrt{x^2 + y^2}$.

Suppose, now, that we apply the following non-linear coordinate transformation to the SHO phase space:

$$x \to x - r = x - \sqrt{x^2 + y^2}\,. \tag{2}$$

This translates all points on a circle of radius r in the positive x-direction by an amount equal to r. A family of circles concentric about the origin in the original space will now share a common tangent point at the origin of the transformed space (see figure 1). The key feature of this transformation is that some subset of the original phase space decreases in topological dimension as a result of the transformation. For the case at hand, the entire x-axis is mapped onto a point. As we shall see, this type of 'infinity-to-one' mapping will be reflected in the behaviour of the transformed dynamical system.

The equation of a transformed circle in the new space is given by

$$(x - r)^2 + y^2 = r^2 \tag{3}$$

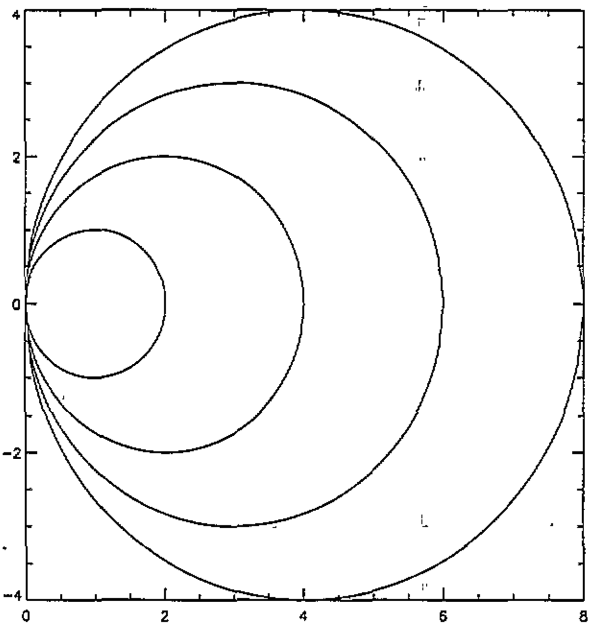

Figure 1. Some examples of circular orbits of different radii, all sharing a common point at the origin.

or, solving for r,

$$r = \frac{1}{2x}(x^2 + y^2) . \tag{4}$$

Using equation (4), we can easily apply the transformation to the SHO. The transformed SHO equations of motion in the new coordinate system are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x = y \qquad \frac{\mathrm{d}}{\mathrm{d}t}y = \frac{y^2}{2x} - \frac{1}{2}x . \tag{5}$$

From the above discussion, the solutions of (5) will be the family of transformed circles, all sharing a common tangent point at the origin. Such intersection of many phase space trajectories is not so unusual. An attracting fixed point, for example, is approached asymptotically for all initial conditions in its basin of attraction (i.e. solutions are unique for finite times). What *is* unusual about (5) is that the common point is intersected in finite time and further is not a fixed point. This is easily seen by taking the limit of (5) along a solution of radius r:

$$\lim_{x,y\to 0} \frac{\mathrm{d}}{\mathrm{d}t}x = 0 \qquad \lim_{x,y\to 0} \frac{\mathrm{d}}{\mathrm{d}t}y = r . \tag{6}$$

Thus, the origin is a singularity of (5), where neither past nor future time evolution is uniquely determined. Henceforth, we shall refer to (5) as the *non-deterministic harmonic oscillator* (NDHO).

The NDHO provides a simple example of the type of system we are examining: solutions of the equations of motion are a family of closed loops ('transients') all sharing a common tangent point. From equations (6), the dynamics of the NDHO are not defined by the equations of motion alone (though this is not a necessary condition for a system to be non-deterministic in the sense described in this paper [9]; more details will be given below). However, let us imagine we have built an NDHO in a laboratory. How would it behave? We note that all *physical* systems are subject to external perturbations, or 'noise'. While the physical state of our NDHO is far (in phase space) from the point (0, 0), external noise will have little effect, provided the average amplitude of the fluctuations is small compared to r for that trajectory. However, as the trajectory approaches the origin, noise plays a larger role. Solutions for all r converge together, ultimately intersecting at $(0, 0)$. Thus, noise will cause the trajectory to jump between solutions of widely differing r in a random way.

What is the effect of this on our laboratory NDHO? Suppose we begin the system on a solution of radius r_1. As we watch the system evolve forward in time, we will find that after it passes near the origin the trajectory has changed to a completely different solution of radius r_2. Repeating the experiment with the same initial conditions, we find that the trajectory jumps to a completely different solution of radius r_3, where $r_3 \neq r_2$. Were we to repeat this a large number of times, for different values of r_1, we would find that the solution after the singularity is completely unrelated to the solution before. If the NDHO were allowed to run for several oscillations, a time series measurement of one variable would appear as a piecewise continuous sequence of oscillations with different amplitudes. Further, the sequence of amplitudes would be random and unpredictable. We term this behaviour *piecewise deterministic dynamics*, since in between jumps the behaviour is essentially deterministic, but changes in an unpredictable way at some particular point along the trajectory.

3. A physically motivated example

Let us now describe a non-deterministic system, based on physical considerations, which exhibits piecewise deterministic dynamical behaviour. This system is a model of the behaviour of neutron star magnetic fields. We describe it briefly; for a more detailed discussion, the reader is referred to [10].

The model envisions two equal but oppositely charged concentric spherical shells which are allowed to rotate differentially. The magnetization of one shell ($\boldsymbol{M}_1$) will interact with the magnetic field of the second shell ($\boldsymbol{H}_2$), as well as experience non-electromagnetic ('mechanical') interactions with the surrounding medium. The magnetic interactions include a term to induce precession of $\boldsymbol{M}_1$ about the instantaneous direction of $\boldsymbol{H}_2$ and the Landau–Lifshitz magnetic damping, which tends to align $\boldsymbol{M}_1$ with the direction of $\boldsymbol{H}_2$. The mechanical interaction is taken as a simple damping, proportional to the difference in angular velocities of the two spheres. Parametrizing the interactions, we obtain the following equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M}_1 = \bar{\gamma}(\boldsymbol{M}_1 \times \boldsymbol{H}_2) - \bar{\lambda}\left(\frac{\boldsymbol{M}_1 \cdot \boldsymbol{H}_2}{M_1^2}\boldsymbol{M}_1 - \boldsymbol{H}_2\right) - \bar{\bar{\eta}} \cdot (\omega_1 - \omega_2). \qquad (7)$$

Following the scaling procedure described in [10] and conserving angular momentum, we arrive at the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{m} = -\boldsymbol{m} \times \hat{\boldsymbol{z}} - \lambda\left(\frac{\boldsymbol{m} \cdot \hat{\boldsymbol{z}}}{m^2}\boldsymbol{m} - \hat{\boldsymbol{z}}\right) - \bar{\bar{\epsilon}}(\boldsymbol{m} - \hat{\boldsymbol{z}}) \qquad (8)$$

where $\boldsymbol{m}$ is the scaled magnetization, τ is the scaled time, λ is the scaled Landau damping parameter and $\bar{\bar{\epsilon}}$ is the scaled viscous damping parameter tensor.

Examination of (8) reveals axial symmetry about the z-axis. This prompts us to make the following transformation:

$$x = \sqrt{{m_x}^2 + {m_y}^2} \qquad z = m_z \qquad \phi = \arctan\frac{m_y}{m_x} \qquad (9)$$

which implies

$$m_x \to x\cos\phi \qquad m_y \to x\sin\phi \qquad m_z \to z\,. \qquad (10)$$

Substituting the above transformations into (8), we obtain

$$\dot{x} = \frac{\lambda x z}{x^2+z^2} - \epsilon x \qquad \dot{z} = \frac{\lambda z^2}{x^2+z^2} - \bar{\epsilon} z - (\lambda - \bar{\epsilon}) \qquad (11)$$

and

$$\dot{\phi} = 1 \qquad (12)$$

where an overdot again represents differentiation with respect to the scaled time τ. The ϕ equation is trivial, simply representing a constant precession about the z-axis. Any interesting dynamical behaviour must occur in (11).

Numerical integration of (11) for $\epsilon < \lambda$ yields the phase space plot in figure 2 and the time series in figure 3. Note the apparent intersection of trajectories near the origin, indicating that something strange is happening in this region. Indeed, it can be shown [9, 11] that for the RHS of (11), the point (0, 0) is a non-Lipschitz singularity. We use the following argument to show that the point (0, 0) is shared by all bounded trajectories (for more details the reader is referred to [9, 11]):

(i) The only fixed point of (11) for $\epsilon < \lambda$ is at (0, 1).

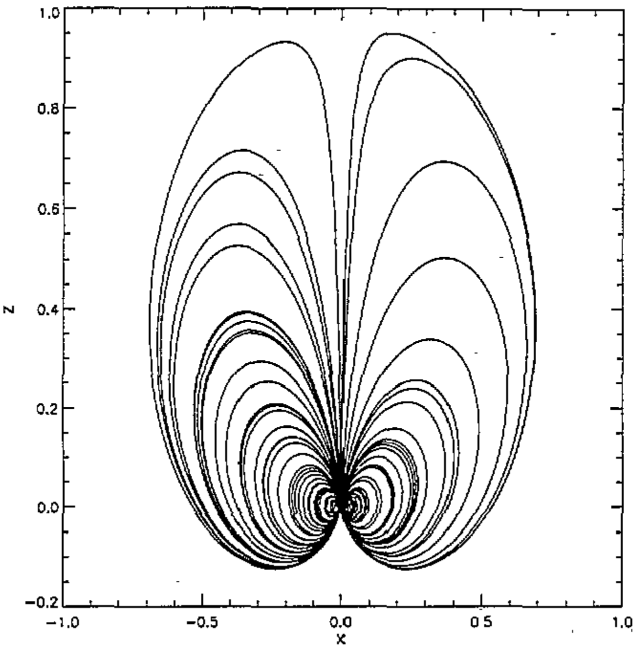

Figure 2. Phase plot of solutions in the neutron star model for $\lambda = 1.0, \epsilon = 0.6, \overline{\epsilon} = 0.7$.

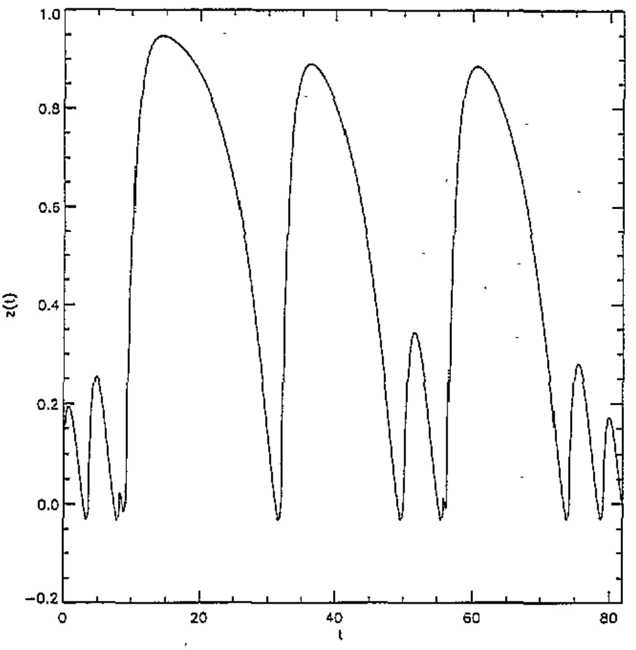

Figure 3. Time series $z(t)$ versus t for the neutron star model with parameter values $\lambda = 1.0, \epsilon = 0.6, \overline{\epsilon} = 0.7$.

(ii) The sole fixed point is a saddle and thus is not an attractor for a set of initial conditions of non-zero measure [12]. This also implies that no periodic orbits exist [12].

(iii) By the Poincaré–Bendixson theorem, if the solutions of a two-degree of freedom dynamical system contain only Lipschitz points, then the only possible bounded asymptotic solutions are stationary or periodic. As neither possibility exists for (11), (0, 0) must be contained by bounded solutions for *all* initial conditions.

At this point, the reader may be concerned about the seemingly unphysical nature of intersecting phase space trajectories. However, we are discussing the behaviour only in the presence of noise. The singularity, being a set of measure zero, is never actually encountered. Even so, the nature of the solutions near the singularity combined with the presence of random perturbations will have a definite effect on the dynamical behaviour. Another interesting technical point is that in the absence of noise equations (11) are actually deterministic. It can be shown [9] that if the singularity is approached along any of the solutions of (11), the derivatives are uniquely defined. This is to be contrasted with the NDHO, for which the value of the RHS of the equations of motion at the singularity could not be uniquely determined, even in the zero-noise limit. Equations (11) are thus not rigorously non-deterministic in the sense of the NDHO. However, the introduction of *any* noise, no matter how small, destroys the determinism of the neutron star model and so in any physical situation, we deem the equations of motion to be effectively non-deterministic at the singularity. Of course, noise itself is stochastic, so rigorously speaking, the preceeding statement applies to all dynamical systems. As we shall see in the following section, what one really should examine is the large-scale time-averaged behaviour in the presence of noise compared to the classical, zero-noise behaviour.

4. Uncertainty in piecewise deterministic dynamics

Let us now take a close look at the effect of uncertainty in a dynamical system in an effort to see what is implied by the existence of the type of non-deterministic singularity we have described. While one often describes systems in the classical domain by deterministic rules, a real system is always subject to uncertainty due to inaccuracy in our measurements, the effect of external perturbations and ultimately quantum mechanical uncertainty. Rather than attempting to account for all of these via one single equation, we divide an experiment into a 'system' (the thing being studied) and 'noise' (everything else). As the 'noise' generally involves many (10^{26}) degrees of freedom and is often attributable to causes outside of our control and knowledge, we are forced to treat the noise in a statistical manner.

In general, we think of a dynamical system as a set of differential equations

$$\dot{x} = f(x) \tag{13}$$

which describe the evolution of the J dependent variables x_j in some J-dimensional phase space (we will confine our discussion to autonomous systems). In purely mathematical terms, each point of the phase space is unique and distinguishable. For a physical system, however, this is not true. The noise implies a minimum uncertainty scale, which we shall call δ. Two points closer than δ will be indistinguishable. Let us then divide up our phase space into many regions of size δ, each with a volume $\propto \delta^J$. We may then think of calculating the average behaviour of (13) in each volume, as well as the expected variance about this average.

Suppose that $f(x)$ is a polynomial, or Taylor expandable within a given volume centred on a point x_0 (all such functions necessarily satisfy the uniqueness theorem). For notational simplicity, we take our example as a two-dimensional system, though the result applies to systems of any dimension. The Taylor series in this case is

$$\begin{aligned} f(x_0+x, y_0+y) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(x,y)|_{x=x_0, y=y_0} \\ &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{02}y^2 + a_{11}xy + \cdots \end{aligned} \tag{14}$$

where a_{ij} represents the series coefficents. The average value of $f(x, y)$ in a given volume is

$$\langle f(x_0 + x, y_0 + y)\rangle = \frac{1}{V}\int_{-\delta/2}^{+\delta/2}\int_{-\delta/2}^{+\delta/2} f(x_0 + x, y_0 + y)\,\mathrm{d}x\,\mathrm{d}y$$
$$V = \int_{-\delta/2}^{+\delta/2}\int_{-\delta/2}^{+\delta/2} \mathrm{d}x\,\mathrm{d}y = \delta^2 \tag{15}$$

where we have chosen the region as a cube of side δ centred on (x_0, y_0), with total volume $V = \delta^2$. Applying this to the series in (14), we find

$$\langle f(x_0 + x, y_0 + y)\rangle = a_{00} + \tfrac{1}{3}\delta^2(a_{20} + a_{02}) + \cdots . \tag{16}$$

Note that terms with odd powers in x or y cancel when integrated over the symmetric interval, thus the average value $\langle f(x_0 + x, y_0 + y)\rangle$ is simply $f(x_0, y_0)$ plus correction terms in even powers of δ. Assuming $\delta \ll 1$, we keep only the leading correction term.

Having found the average value of $f(x, y)$ in a cell, we now wish to find the uncertainty in this value and especially how this uncertainty relates to the fundamental uncertainty scale δ in the phase space. The root mean square (RMS) of $f(x, y)$ over a cell is simply

$$\begin{aligned}\sigma &= (\langle f(x_0 + x, y_0 + y)^2\rangle - \langle f(x_0 + x, y_0 + y)\rangle^2)^{1/2} \\ &= \delta(\tfrac{1}{3}(a_{10}^2 + a_{01}^2))^{1/2} .\end{aligned} \tag{17}$$

Not surprisingly, we find that for 'nice' functions, the uncertainty (to leading order) in the dynamical vector field goes like the inherent uncertainty in the phase space. Thus in the 'classical' or 'thermodynamic' limit, where δ is taken as very small, we find that the dynamics is essentially unchanged. The result applies even for deterministic chaos. The Lorenz equations, for example, are polynomial in the phase space variables. In the chaotic parameter regime, the non-linearity acts to spread any initial uncertainty across the strange attractor. However, this is (loosely speaking) a global property of the system. Locally, the dynamical uncertainty is related only to the uncertainty in the phase space variables.

What happens when we apply the above analysis to the types of systems we are discussing in this paper? Both the NDHO and the neutron star model contain essential singularities and hence cannot be Taylor expanded around these points. However, we can explicitly calculate the average value and variance of the equations of motion in a phase space cell about the singularity. Consider the neutron star model. First, we note that because of the nature of the singularity, the quantities obtained for the various integrals will depend on the order of integration. This is easily dealt with by transforming to plane polar coordinates $(x, z) = (r\sin\theta, r\cos\theta)$ and taking the phase space cell as a disc centred on the singularity. Performing the integration over this cell, we find that the average values are well defined:

$$(\langle\dot{x}\rangle, \langle\dot{z}\rangle) = (0, \epsilon - \tfrac{1}{2}) . \tag{18}$$

The RMS deviations from these values are

$$\sigma_x = \sigma_z = \sqrt{\frac{1 + 2\epsilon^2\delta^2}{8}} \tag{19}$$

and we now begin to see the fundamental difference between 'smooth' deterministic and piecewise deterministic dynamics. In the classical limit, the uncertainties in (19) do not become arbitrarily small! As $\delta \to 0$, we find that the uncertainty in the RHS of (11) goes to $1/2\sqrt{2}$, which is of the same order as $(\langle\dot{x}\rangle, \langle\dot{z}\rangle)$. In the presence of *any* uncertainty, regardless of how small, the equations of motion do not even approximately determine the behaviour near the singularity. This is why we term the dynamics described here as *non-deterministic.*

Having established this key distinction, let us briefly compare deterministic chaos and piecewise deterministic dynamics. Again, the action of a deterministic chaotic system is to spread an initial uncertainty over a larger and larger portion of the strange attractor as time goes by. This spreading is caused by a global instability, associated with one or more positive Lyapunov exponents [13]. For piecewise deterministic dynamics, the uncertainty is essentially induced at a particular point, or more precisely, in some small localized region. The effect of this can be illustrated by using the information approach first suggested by Shaw [14]. Assuming that the amount of information contained in some region of phase space is proportional to the volume of the cell, the rate of information generation is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} \tag{20}$$

where the RHS is the *Lie derivative* of the system, defined as

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \sum \frac{\partial_i \dot{x}_i}{\partial x_i}\,. \tag{21}$$

The total change in the system information may be found as the integral of the Lie derivative w.r.t. time along the system trajectory. For a deterministic chaotic system, information is created at a rate proportional to the largest positive Lyapunov exponent. Noting that an increase in system information corresponds to a decrease in the knowledge of an observer, the phenomenon of deterministic chaos implies a steady decrease of an observer's knowledge of the system's past, e.g. its initial conditions. Another way of looking at this is to say that nearby points on a trajectory have a high degree of correlation, while this correlation decreases as the points become more separated. Indeed, this property may be utilized when attempting a time series reconstruction of a chaotic system [15].

For the piecewise deterministic case, we examine the NDHO, as we know the solutions analytically. Taking the initial conditions to be $(x, y) = (0, 0)$ at $t = 0$, the solutions of (5) are

$$x = A(1 - \cos t) \qquad y = A\sin t\,. \tag{22}$$

The reader may, at this point, be concerned that we give the initial conditions at the singularity. We justify this by noting that with regard to information calculations, we can only talk about those points which are *distinguishable.* Since there will always be some uncertainty in the system, we are really referring to some set of points near the origin, with the initial conditions being somewhere within this region. The Lie derivative of the NDHO is given by

$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{y}{x}\,. \tag{23}$$

Integrating along a solution over half a cycle, we find the total information generated to be

$$\Delta H = \int_0^{\pi} \frac{\sin t}{1 - \cos t} = \log(1 - \cos t)|_0^{\pi} = \infty\,. \tag{24}$$

The gain in system information is infinite, which means that the observer has zero knowledge of the system's past. This is another key feature of piecewise deterministic dynamics: whenever a trajectory passes near the singularity, the future time evolution is completely decoupled from the past.

5. Piecewise determinism and predictability

As shown above, piecewise deterministic dynamics has the property of being predictable for short times (between intersections of the singularity), yet completely unpredictable over long time periods. Long-term unpredicability is also one of the hallmarks of deterministic chaos, but it is here that the similarity ends. Aside from being described by deterministic equations, deterministic chaos is often characterized by exponential divergence of initially close solutions and associated with a complex fractal structure, the strange attractor. Non-deterministic chaos derives its unpredictability from a more violent, but localized instability. Further, there exists no attractor, strange or otherwise, at least in the usual sense of the word.

Formally, we can study the effect of noise on a 'classical' system by constructing a *Langevin equation*,

$$\dot{x} = f(x) + \epsilon\phi(t) \tag{25}$$

where $f(x)$ represents the classical part of the equation of motion, $\phi(t)$ is some random function with mean zero and standard deviation one which reflects the statistics of the noise and ϵ controls the average size of the random perturbations. The trajectory in phase space indicated by (25) is a Brownian motion and as such there exists a time-dependent probability density of finding the system state at some particular point in phase space. Any initial probability density will tend to diffuse through phase space in a manner governed by the forward Kolmogorov equation [16]. The rate of this diffusion is directly related to the rate at which the system generates information, which is a measure of the system's predictability.

For the systems we have considered, the classical part of the Langevin equation is singular, and so the associated forward Kolmogorov equation will also be singular. The analysis in the previous section indicates that we expect an explosion of information at the singularity and this can be illustrated by integrating the Langevin equation for the neutron star model for some set of initial conditions confined to a small region of phase space. For this simulation, we took $\phi(t)$ to be distributed as a Gaussian and ϵ to be of the order of the integration step. Figure 4 illustrates the generation of information at the singularity due to the presence of external noise. Initially, we see smooth deformation of the initial set of points. However, once the singularity is encountered, points are scattered and soon are randomly spread through a region of phase space (in this case, the region is enclosed by the homoclines of the saddle at (0, 1) [9]). This behaviour is in stark contrast to what one expects from deterministic chaos, where an initial volume is stretched and folded [13], spreading across the attractor in a smooth fashion. Dynamical measures such as the Lyapunov exponent are meaningless. The 'attractor' exists only in a statistical sense, representing the probability density that a particular point in phase space will be visited. This distribution is shown in figure 5, and was calculated by integrating the Langevin equation for 200 million time steps.

Figure 5 gives a long-term, global statistical picture. However, the simple structure of the solutions near the singularity allows us to easily extract more useful statistical information. In particular, we shall utilize the fact that away from the singular point, the dynamics is quite well behaved. Let us return to the NDHO, as its solutions are known analytically. The solutions of the NDHO may be parametrized by their radius. Solutions away from the singularity have essentially constant radii; the big jumps occur only near the singularity. Can we predict the probability that a circle of given r is chosen when the orbit leaves some neighbourhood about the origin? Let us define this neighbourhood as a disc of radius δ and note that an orbit leaving this neighbourhood does so with angle θ, which we

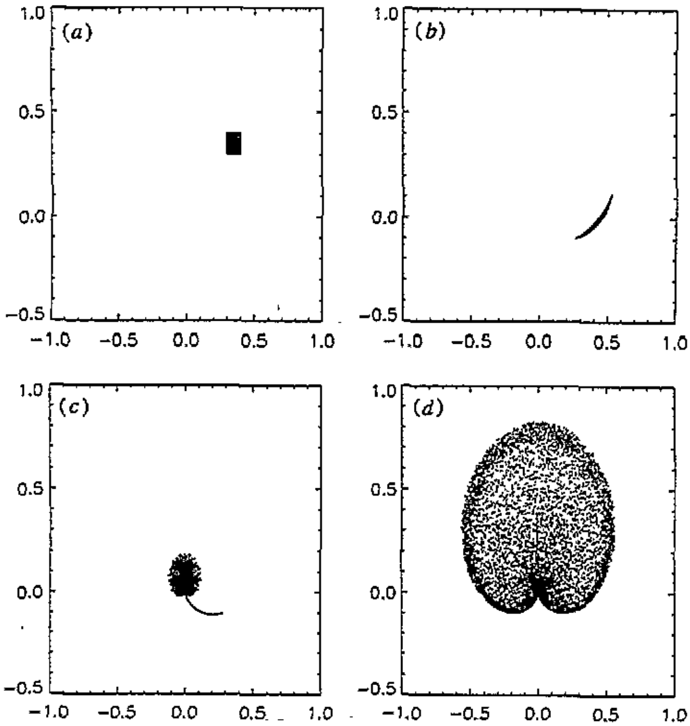

Figure 4. Loss of information due to the singularity. (*a*) 10 000 initial points are arranged in a 100×100 square. (*b*) Initial evolution. (*c*) When the singularity is encountered, initally close points are scattered randomly. (*d*) All information about the initial conditions has been lost. The only information carried by the system is in the density of trajectories.

take as measured from the y-axis. Now, assuming the external fluctuations to be isotropic, the probability density of picking a particular θ is constant, i.e.

$$p(\theta)\,\mathrm{d}\theta \propto \mathrm{d}\theta\,. \tag{26}$$

Next, we note that everywhere except at the origin the existence and uniqueness theorem applies to solutions of (5), thus each circle of radius r is associated with a unique θ and we may write θ as a function of r. Substituting into $p(\theta)$, we find

$$p(r)\,\mathrm{d}r \propto \frac{\partial\theta(r)}{\partial r}\,\mathrm{d}r\,. \tag{27}$$

The probability of getting a circle between r and $r + \Delta r$ is simply

$$P(r, \Delta r) \propto \int_r^{r+\Delta r} \frac{\partial\theta(r)}{\partial r}\,\mathrm{d}r = \theta(r+\Delta r) - \theta(r)\,. \tag{28}$$

For the case at hand, we find

$$P(r, \Delta r) \propto \arccos\frac{\delta}{2(r+\Delta r)} - \arccos\frac{\delta}{2r}\,. \tag{29}$$

This approach (first described in [11]) is somewhat simplified. A rigorous derivation would account for the statistical properties of the noise and derive $P(r, \Delta r)$ via stochastic calculus. The above does show, however, that the simple structure of the solutions of a non-deterministic system lends itself to the construction of statistical arguments. Further, with

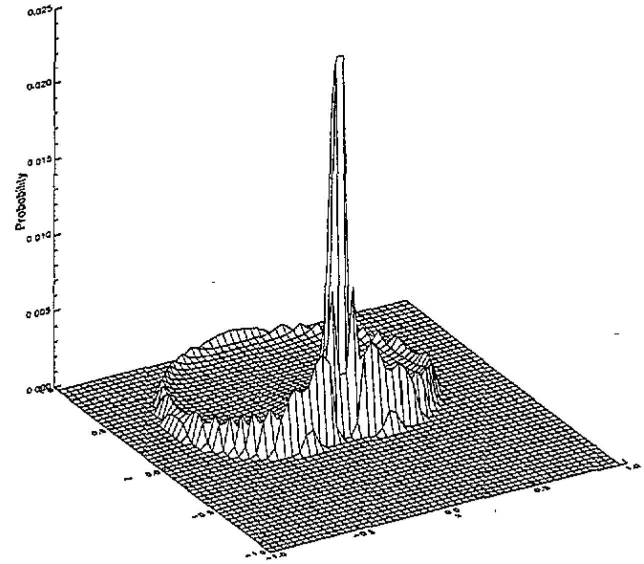

Figure 5. Probability of finding the system in a particular region of phase space. The distribution was found by integrating the equations for 200 million steps and totalling the amount of time spent in each region.

a judicious choice of δ, based on knowledge of the average amplitude of the fluctuations, the above procedure should yield a good approximation of the true distribution $P(r, \Delta r)$.

6. Controlling piecewise deterministic dynamics

The control of deterministic chaotic systems using small perturbations has been a subject of recent vigorous research [15]. The most popular method of controlling deterministic chaos involves the stabilization of (otherwise) unstable periodic orbits which are embedded in the chaotic motion. As there exist an infinity of orbits, a rich variety of behaviours may be extracted from the controlled deterministic chaotic system, allowing for flexibility and easy optimization of a system's behaviour.

For a piecewise deterministic system, we have a similar situation. With a continuum of different solutions intersecting at a single point, we can easily effect control via an appropriate perturbation. Similar to the previous section, we simply examine how solutions leave a δ-neighbourhood about the singularity. Again, away from the singularity the solution is well defined. Suppose that each different solution may be parametrized by some quantity γ *(in the case of the NDHO, this is the radius)*. A given solution, parametrized by γ_0, will intersect the δ-neighbourhood at a unique point (x_0, y_0). From this, we may construct the angle $\theta(\gamma_0) = \arctan x_0/y_0$.

The angle $\theta(\gamma)$ we term the *control angle* and the reason should be obvious. To keep the system on a solution with parameter γ_0, we need only wait until the trajectory approaches the origin and then perturb it so that it leaves at angle $\theta(\gamma_0)$. This perturbation will be quite small, of the order of δ, with the size of δ being determined largely by the noise amplitude. We see that in a piecewise deterministic system, there is a continuum of possibilities available through small control perturbations. If a change in system behaviour were required, it is easily and quickly effected by simply changing $\theta(\gamma)$. In fact, one could

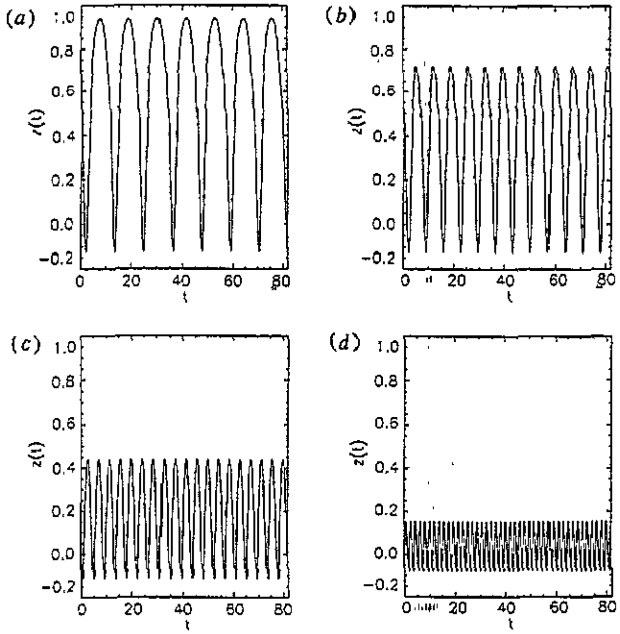

Figure 6. Examples of output from the neutron star model when the control algorithm is applied. Signals are shown for (*a*) $\theta(\gamma) = 0.005$, (*b*) 0.03, (*c*) 0.08 and (*d*) 0.2.

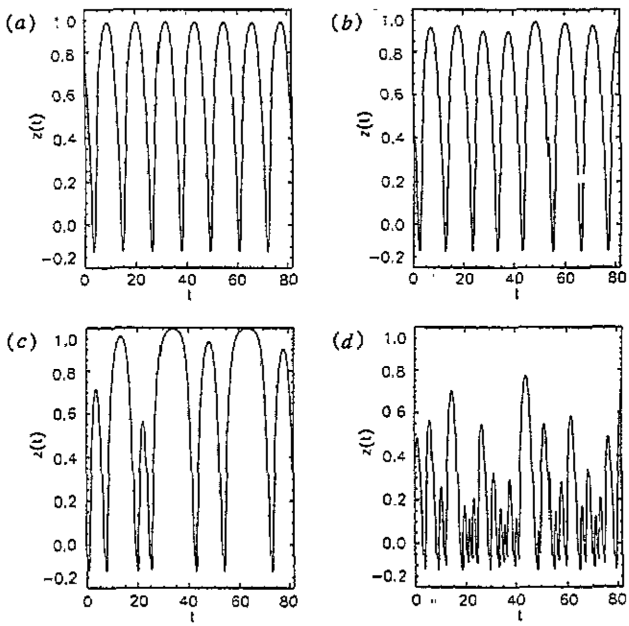

Figure 7. The control algorithm begins to break down if δ is chosen to be comparable to the noise level. Signals are shown for (*a*) $\delta = 10^4\sigma$, (*b*) $10^3\sigma$, (*c*) $10^2\sigma$, and (*d*) 10σ, where σ is the RMS of the noise.

vary $\theta(\gamma)$ as a function of time to induce arbitrarily complex behaviour.

As an example, we have applied this control algorithm to (11). To simulate the effect of noise, a small (10^{-4} of the integration stepsize) normally distributed random number was added at each integration step. The controlled signals for various values of θ are shown in figure 6. Figure 7 shows the effect of noise for different values of δ.

7. Discussion

The type of 'non-determinism' described above should not be construed as implying complete stochasticity. Indeed, the behaviour of both the NDHO and neutron star model are uniquely determined away from the singularity. It is at this point, and this point only, that the non-deterministic nature of the equations arises. In the presence of random fluctuations, which are ubiquitous (though perhaps small) in physical systems, the non-determinism, albeit at a single point, becomes important. The resulting dynamics consists of a random sequence of 'transient' oscillations.

As yet, there exists no firm evidence that the behaviour described here occurs in nature (although some indications exists from studies of biological systems [8]). However, application of several standard measures (power spectrum, Lyapunov exponent, etc) to a time series generated by (11) would lead one to believe that one is examining an instance of deterministic chaos [9], so the lack of evidence to date places little constraint on the question. As the singular behaviour occurs only at a single point, discriminating piecewise deterministic from chaotic dynamics is a difficult task, though work is progressing in this area. However, the ability to make such a distinction may prove important, as issues of prediction and control would be addressed much differently for a non-deterministic chaotic system. Crutchfield has shown that in the context of model building, assuming determinism when the underlying process is non-deterministic leads to undue complexity in the model [17]. It would seem reasonable to search for piecewise deterministic dynamics in apparently complex systems, especially in cases where traditional analysis tools (which, again, assume determinism) have failed. Further, it may be possible to construct non-deterministic systems and exploit the ability to control them via small perturbations.

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